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Global Smoothness and Uniform Convergence of Smooth Picard Singular Operators

G. A. ANASTASSIOU
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

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Abstract—In this article, we continue with the study of smooth Picard singular integral operators over the real line regarding their simultaneous global smoothness preservation property with respect to the L_p norm, $1 \le p \le \infty$, by involving higher-order moduli of smoothness. Also, we study their simultaneous approximation to the unit operator with rates involving the modulus of continuity with respect to the uniform norm. The produced Jackson type inequalities are almost sharp, containing elegant constants, and they reflect the high order of differentiability of the engaged function. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Simultaneous global smoothness, Simultaneous approximation, Picard singular integral, Modulus of smoothness, Rate of convergence.

1. INTRODUCTION

The global smoothness preservation property of singular integrals has been studied initially in [1] and later in [2]. The rate of convergence of singular integrals has been studied initially in [3–5], later in [6–8], and most recently was studied in detail in [9,10] over the real line, just for the Picard general type integral operators case. All the above-mentioned papers, along with the earlier ones [11,12] by the author, motivate the current work.

More precisely here, we continue with the study of smooth Picard singular integral operators over $\mathbb{R}$ acting on highly smooth functions. First, we study their simultaneous global smoothness preservation property with respect to $\|\cdot\|_p$, $1 \le p \le \infty$, by using higher-order moduli of smoothness. Then, we study their simultaneous pointwise and uniform approximation to the unit operator with rates by using the first modulus of continuity. The established estimates are almost optimal and contain nice constants. The modulus of continuity in the estimates is with respect to the higher-order derivative of the engaged function. The discussed operators are not, in general, positive.

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2. GLOBAL SMOOTHNESS PRESERVATION RESULTS

Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function and consider the Lebesgue integral,

$$P_\xi(f;x) := \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+t)\, e^{-|t|/\xi}\, dt, \qquad \xi > 0, \quad x \in \mathbb{R}. \tag{1}$$

We would like to mention from [2, pp. 407–412], the following result regarding global smoothness preservation properties of P_ξ, see there (16.23), (16.36), (16.48).

THEOREM 1. *Let* $h > 0$.

(i) *Assume that* $\omega_m(f,h) < \infty$ *and* $P_\xi(f;x) \in \mathbb{R}$, *then,*

$$\omega_m(P_\xi f, h) \le \omega_m(f,h). \tag{2}$$

Inequality (2) is sharp, namely it is attained by $f(x) = x^m$.

(ii) *Let* $f \in L_1(\mathbb{R})$, *then*

$$\omega_m(P_\xi f, h)_1 \le \omega_m(f,h)_1. \tag{3}$$

And

(iii) *Let* $f \in L_p(\mathbb{R})$, p, $q > 1$, *such that* $1/p + 1/q = 1$, *then,*

$$\omega_m(P_\xi f, h)_p \le \frac{2}{p^{1/p} q^{1/q}}\omega_m(f,h)_p. \tag{4}$$

Above, we use for $m \in \mathbb{N}$, the m^{th} modulus of smoothness for $1 \le p \le \infty$,

$$\omega_m(f,h)_p := \sup_{0\le t\le h} \left|\Delta_t^m f(x)\right|_{p,x}, \tag{5}$$

where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j} f(x+jt), \tag{6}$$

see also [13, p. 44]. Denote $\omega_m(f,h)_\infty = \omega_m(f,h)$. In [9,10], we studied extensively the convergence properties to the unit of the following smooth Picard singular integral operator $P_{r,\xi}(f;x)$ defined next.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1,\dots,r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{7}$$

that is, $\sum_{j=0}^{r} \alpha_j = 1$.

We consider the Lebesgue integral,

$$P_{r,\xi}(f;x) := \frac{1}{2\xi}\int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt)\right) e^{-|t|/\xi}\, dt. \tag{8}$$

Operators $P_{r,\xi}$ are not positive, see [9]. We notice that $1/2\xi \int_{-\infty}^{\infty} e^{-|t|/\xi}\, dt = 1$. We observe for $j = 1,\dots,r$ that

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+jt)\, e^{-|t|/\xi}\, dt = P_{\xi j}(f;x). \tag{9}$$

Furthermore, it holds

$$P_{r,\xi}(f;x) = \alpha_0 f(x) + \sum_{j=1}^{r} \alpha_j P_{\xi j}(f;x). \tag{10}$$

Notice that $P_{1,\xi} = P_\xi$. Assuming $P_{\xi j}(f;x) \in \mathbb{R}$, $j = 1,\dots,r$, clearly, one sees that $P_{r,\xi}(f;x) \in \mathbb{R}$.

The following global smoothness result holds.

THEOREM 2. *Let $h > 0$, $f : \mathbb{R} \to \mathbb{R}$.*

(i) *Assume $P_{\xi j}(f;x) \in \mathbb{R}$, all $j = 1, \dots, r$, $\xi > 0$, $x \in \mathbb{R}$, and $\omega_m(f,h) < \infty$. Then,*

$$\omega_m\left(P_{r,\xi} f, h\right) \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f, h\right). \tag{11}$$

(ii) *Assume $f \in L_1(\mathbb{R})$, then*

$$\omega_m\left(P_{r,\xi} f, h\right)_1 \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f, h\right)_1. \tag{12}$$

(iii) *Assume $f \in L_p(\mathbb{R})$, p, $q > 1$, such that $1/p + 1/q = 1$. Then,*

$$\omega_m\left(P_{r,\xi} f, h\right)_p \leq \frac{2\left(\sum\limits_{j=0}^{r} |\alpha_j|\right)}{p^{1/p} q^{1/q}} \omega_m\left(f, h\right)_p. \tag{13}$$

PROOF.

(i) See that

$$\begin{aligned}
\omega_m\left(P_{r,\xi} f, h\right) &\overset{(10)}{=} \omega_m\left(a_0 f(x) + \sum_{j=1}^{r} \alpha_j P_{\xi j}(f;x), h\right) \\
&\leq |\alpha_0|\, \omega_m(f,h) + \sum_{j=1}^{r} |\alpha_j|\, \omega_m\left(P_{\xi j} f, h\right) \\
&\overset{(2)}{\leq} |\alpha_0|\, \omega_m(f,h) + \left(\sum_{j=1}^{r} |\alpha_j|\right) \omega_m(f,h) = \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m(f,h).
\end{aligned}$$

That is proving (11).

(ii) Next, we observe

$$\begin{aligned}
\omega_m\left(P_{r,\xi} f, h\right)_1 &\overset{(10)}{=} \omega_m\left(\alpha_0 f(x) + \sum_{j=1}^{r} \alpha_j P_{\xi j}(f;x), h\right)_1 \\
&\leq |\alpha_0|\, \omega_m(f,h)_1 + \sum_{j=1}^{r} |\alpha_j|\, \omega_m\left(P_{\xi j} f, h\right)_1 \\
&\overset{(3)}{\leq} |\alpha_0|\, \omega_m(f,h)_1 + \left(\sum_{j=1}^{r} |\alpha_j|\right) \omega_m(f,h)_1 = \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m(f,h)_1.
\end{aligned}$$

That is proving (12).

(iii) Finally, we have

$$\begin{aligned}
\omega_m\left(P_{r,\xi} f, h\right)_p &\overset{(10)}{=} \omega_m\left(\alpha_0 f(x) + \sum_{j=1}^{r} \alpha_j P_{\xi j}(f;x), h\right)_p \\
&\leq |\alpha_0|\, \omega_m(f,h)_p + \sum_{j=1}^{r} |\alpha_j|\, \omega_m\left(P_{\xi j} f, h\right)_p \\
&\overset{(4)}{\leq} |\alpha_0|\, \omega_m(f,h)_p + \left(\sum_{j=1}^{r} |\alpha_j|\right) \frac{2}{p^{1/p} q^{1/q}} \omega_m(f,h)_p =: (*).
\end{aligned}$$

However, it holds that

$$1 \leq \frac{2}{p^{1/p} q^{1/q}},$$

by Corollary 13.3 [14, p. 190]. Hence, we find

$$\begin{aligned}(*) &\leq |\alpha_0| \frac{2}{p^{1/p} q^{1/q}} \omega_m (f,h)_p + \left(\sum_{j=1}^{r} |\alpha_j| \right) \frac{2}{p^{1/p} q^{1/q}} \omega_m (f,h)_p \\ &= \frac{2 \left(\sum_{j=0}^{r} |\alpha_j| \right)}{p^{1/p} q^{1/q}} \omega_m (f,h)_p .\end{aligned}$$

That is establishing (13). ∎

Next, we discuss about the derivatives of $P_\xi(f;x)$ and $P_{r,\xi}(f;x)$ and their impact to simultaneous global smoothness preservation and convergence of these operators.

For the next differentiation result we use Theorem 24.5 [15, pp. 193–194] and then the proof is easy.

THEOREM 3. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n \in \mathbb{N}$. Furthermore, suppose that $f^{(j)}(t)e^{-|t|} \in L_1(\mathbb{R})$, for all $j = 0, 1, \ldots, n-1$. Assume that there exist $g_j \geq 0$, $j = 1, 2, \ldots, n$, $g_j \in L_1(\mathbb{R})$, such that, for each $x \in \mathbb{R}$, we have*

$$\left| f^{(j)} (x+t) \right| e^{-|t|} \leq g_j (t), \tag{14}$$

for almost all $t \in \mathbb{R}$, all $j = 1, 2, \ldots, n$. Then, $f^{(j)}(x+t)e^{-|t|}$ defines a Lebesgue integrable function with respect to t, for each $x \in \mathbb{R}$, all $j = 1, \ldots, n$, and

$$\left(\int_{-\infty}^{\infty} f(x+t)\, e^{-|t|}\, dt \right)^{(j)} = \int_{-\infty}^{\infty} f^{(j)} (x+t)\, e^{-|t|}\, dt, \tag{15}$$

for all $x \in \mathbb{R}$, all $j = 1, \ldots, n$.

We apply the last theorem to our case. First comes the related differentiation result about operator P_ξ.

THEOREM 4. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n \in \mathbb{N}$. Furthermore, suppose that $f^{(j)}(t)e^{-|t|/\xi} \in L_1(\mathbb{R})$, for all $j = 0, 1, 2, \ldots, n-1$, $\xi > 0$. Assume that there exist $g_{j,\xi} \geq 0$, $j = 1, 2, \ldots, n$, $g_{j,\xi} \in L_1(\mathbb{R})$, such that, for each $x \in \mathbb{R}$, we have*

$$\left| f^{(j)} (x+t) \right| e^{-|t|/\xi} \leq g_{j,\xi} (t), \tag{16}$$

for almost all $t \in \mathbb{R}$, all $j = 1, 2, \ldots, n$. Then, $f^{(j)}(x+t)e^{-|t|/\xi}$ defines a Lebesgue integrable function with respect to t, for each $x \in \mathbb{R}$, all $j = 1, \ldots, n$, and

$$\left(P_\xi (f; x) \right)^{(j)} = P_\xi \left(f^{(j)}; x \right), \tag{17}$$

for all $x \in \mathbb{R}$, all $j = 1, \ldots, n$.

PROOF. This is the same as in Theorem 3. ∎

It follows the related differentiation result about $P_{r,\xi}$ operator.

THEOREM 5. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n \in \mathbb{N}$, $r \in \mathbb{N}$. Furthermore, suppose that $f^{(i)}(t)e^{-|t|/r\xi} \in L_1(\mathbb{R})$, for all $i = 0, 1, 2, \dots, n-1$, $\xi > 0$. Assume that there exist $g_{i,r\xi} \geq 0$, $i = 1, 2, \dots, n$, $g_{i,r\xi} \in L_1(\mathbb{R})$, such that, for each $x \in \mathbb{R}$, we have*

$$\left| f^{(i)}(x+t) \right| e^{-|t|/r\xi} \leq g_{i,r\xi}(t), \tag{18}$$

for almost all $t \in \mathbb{R}$, all $i = 1, 2, \dots, n$. Then, $f^{(i)}(x+t)e^{-|t|/j\xi}$ defines a Lebesgue integrable function with respect to t, for each $x \in \mathbb{R}$, all $i = 1, \dots, n$; $j = 1, \dots, r$, and

$$(P_{r,\xi}(f;x))^{(i)} = P_{r,\xi}\left(f^{(i)}, x\right), \tag{19}$$

for all $x \in \mathbb{R}$, all $i = 1, \dots, n$.

PROOF. This is proven by Theorem 4 and (10). ∎

Using Theorems 1 and 4, we obtain the following simultaneous global smoothness result.

THEOREM 6. *Let $h > 0$ and assumptions of Theorem 4 valid.*

(i) *Assume that $\omega_m\left(f^{(i)}, h\right) < \infty$, all $i = 0, 1, \dots, n$, then*

$$\omega_m\left(\left(P_\xi f\right)^{(i)}, h\right) \leq \omega_m\left(f^{(i)}, h\right), \tag{20}$$

for all $i = 0, 1, \dots, n$.

(ii) *Let $f^{(i)} \in L_1(\mathbb{R})$, $i = 0, 1, \dots, n$, then*

$$\omega_m\left(\left(P_\xi f\right)^{(i)}, h\right)_1 \leq \omega_m\left(f^{(i)}, h\right)_1, \tag{21}$$

for all $i = 0, 1, \dots, n$. And

(iii) *Let $f^{(i)} \in L_p(\mathbb{R})$, $i = 0, 1, \dots, n$, p, $q > 1$, such that $1/p + 1/q = 1$, then,*

$$\omega_m\left(\left(P_\xi f\right)^{(i)}, h\right)_p \leq \frac{2}{p^{1/p} q^{1/q}} \omega_m\left(f^{(i)}, h\right)_p, \tag{22}$$

for all $i = 0, 1, \dots, n$.

Using Theorems 2 and 5, we get the more general simultaneous global smoothness result.

THEOREM 7. *Let $h > 0$ and assumptions of Theorem 5 valid.*

(i) *Assume that $\omega_m(f^{(i)}, h) < \infty$, all $i = 0, 1, \dots, n$, then*

$$\omega_m\left(\left(P_{r,\xi} f\right)^{(i)}, h\right) \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right), \tag{23}$$

for all $i = 0, 1, \dots, n$.

(ii) *Let $f^{(i)} \in L_1(\mathbb{R})$, $i = 0, 1, \dots, n$ then,*

$$\omega_m\left(\left(P_{r,\xi} f\right)^{(i)}, h\right)_1 \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right)_1, \tag{24}$$

for all $i = 0, 1, \dots, n$, and

(iii) *Let $f^{(i)} \in L_p(\mathbb{R})$, $i = 0, 1, \dots, n$, p, $q > 1$, such that $1/p + 1/q = 1$, then*

$$\omega_m\left(\left(P_{r,\xi} f\right)^{(i)}, h\right)_p \leq \frac{2\left(\sum_{j=0}^{r} |\alpha_j|\right)}{p^{1/p} q^{1/q}} \omega_m\left(f^{(i)}, h\right)_p, \tag{25}$$

for all $i = 0, 1, \dots, n$.

3. CONVERGENCE RESULTS

Here, let $f \in C^n(\mathbb{R})$ with $\omega_1(f^{(n)}, h) < \infty$, $h > 0$, $n \in \mathbb{N}$. Assume $P_{\xi j}(f;x) \in \mathbb{R}$, for $j = 1, \ldots, r \in \mathbb{N}$, $\xi > 0$, all $x \in \mathbb{R}$. From (10), we obtain

$$P_{r,\xi}(f;x) - f(x) = \sum_{j=1}^{r} \alpha_j \left(P_{\xi j}(f;x) - f(x)\right) \tag{26}$$

and

$$|P_{r,\xi}(f;x) - f(x)| \le \sum_{j=1}^{r} |\alpha_j| \, |P_{\xi j}(f;x) - f(x)|. \tag{27}$$

Here, we have

$$P_{\xi j}(f;x) = \frac{1}{2\xi j} \int_{-\infty}^{\infty} f(x+t) \, e^{-|t|/\xi j} \, dt. \tag{28}$$

We call

$$\Delta_{\xi j}(f;x) := P_{\xi j}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x) (\xi j)^{2m}, \tag{29}$$

$j = 1, \ldots, r$, where $\lfloor \cdot \rfloor$ is the integral part of the number, $x \in \mathbb{R}$.

In (29), the sum collapses when $n = 1$. Clearly, we have

$$\Delta_{\xi}(f;x) = P_{\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x) \xi^{2m}, \qquad x \in \mathbb{R}. \tag{30}$$

We call also

$$\delta_{2m} := \sum_{j=1}^{r} \alpha_j j^{2m} \tag{31}$$

and

$$E_{r,\xi}(f;x) := P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x) \delta_{2m} \xi^{2m}, \qquad x \in \mathbb{R}. \tag{32}$$

We notice that

$$E_{r,\xi}(f;x) = \sum_{j=1}^{r} \alpha_j \Delta_{\xi j}(f;x) \tag{33}$$

and

$$|E_{r,\xi}(f;x)| \le \sum_{j=1}^{r} |\alpha_j| \, |\Delta_{\xi j}(f;x)|, \qquad x \in \mathbb{R}. \tag{34}$$

Here, we study the convergence of operators $P_{r,\xi}$ to the unit operator I with rates, $r \in \mathbb{N}$. First, we present the following.

THEOREM 8. *It holds*

$$|\Delta_{\xi j}(f;x)| \le (\xi j)^n \left[j + \frac{1}{2} + \frac{1}{8j}\right] \omega_1\left(f^{(n)}, \xi\right), \qquad j = 1, \ldots, r, \quad \xi > 0, \tag{35}$$

and

$$|\Delta_{\xi}(f;x)| \le \frac{13}{8} \xi^n \omega_1\left(f^{(n)}, \xi\right). \tag{36}$$

That is, we have

$$\|\Delta_{\xi j}(f)\|_{\infty} \le (\xi j)^n \left[j + \frac{1}{2} + \frac{1}{8j}\right] \omega_1\left(f^{(n)}, \xi\right) \tag{37}$$

and

$$\|\Delta_{\xi}(f)\|_{\infty} \leq \frac{13}{8}\xi^n \omega_1\left(f^{(n)}, \xi\right). \tag{38}$$

PROOF. Here, let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$. By Taylor's formula, see Lemma 2, (2) [16, p. 2], we have

$$f(x+t) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} t^k + \mathcal{R}_n(f; x, x+t), \tag{39}$$

where

$$\mathcal{R}_n(f; x, x+t) := \frac{1}{(n-1)!} \int_x^{x+t} \left(f^{(n)}(s) - f^{(n)}(x)\right)(x+t-s)^{n-1}\, ds, \tag{40}$$

for all x, $t \in \mathbb{R}$.

Using Theorem 6, (14) [16, p. 4], we find

$$|\mathcal{R}_n(f; x, x+t)| \leq \omega_1\left(f^{(n)}, \xi\right)\left[\frac{|t|^{n+1}}{(n+1)!\xi} + \frac{|t|^n}{2n!} + \frac{\xi\, |t|^{n-1}}{8(n-1)!}\right], \tag{41}$$

all $t \in \mathbb{R}$, $\xi > 0$, $j = 1, \dots, r$. From (39), we have

$$f(x+t) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} t^k = \mathcal{R}_n(f; x, x+t) \tag{42}$$

and

$$\begin{aligned} &\frac{1}{2\xi j}\int_{-\infty}^{\infty} f(x+t)\, e^{-|t|/\xi j}\, dt - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}\frac{1}{2\xi j}\int_{-\infty}^{\infty} t^k e^{-|t|/\xi j}\, dt \\ &\quad = \frac{1}{2\xi j}\int_{-\infty}^{\infty} \mathcal{R}_n(f; x, x+t)\, e^{-|t|/\xi j}\, dt. \end{aligned} \tag{43}$$

That is,

$$\begin{aligned} &P_{\xi j}(f; x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x)\,(\xi j)^{2m} \\ &\quad = \frac{1}{2\xi j}\int_{-\infty}^{\infty} \mathcal{R}_n(f; x, x+t)\, e^{-|t|/\xi j}\, dt, \end{aligned} \tag{44}$$

i.e., by (29), we find

$$\Delta_{\xi j}(f; x) = \frac{1}{2\xi j}\int_{-\infty}^{\infty} \mathcal{R}_n(f; x, x+t)\, e^{-|t|/\xi j}\, dt, \tag{45}$$

all $x \in \mathbb{R}$.

Furthermore, we have

$$\begin{aligned} |\Delta_{\xi j}(f; x)| &\leq \frac{1}{2\xi j}\int_{-\infty}^{\infty} |\mathcal{R}_n(f; x, x+t)|\, e^{-|t|/\xi j}\, dt \\ &\overset{(41)}{\leq} \frac{\omega_1\left(f^{(n)}, \xi\right)}{2\xi j}\int_{-\infty}^{\infty}\left[\frac{|t|^{n+1}}{(n+1)!\xi} + \frac{|t|^n}{2n!} + \frac{\xi\,|t|^{n-1}}{8(n-1)!}\right] e^{-|t|/\xi j}\, dt \\ &= \omega_1\left(f^{(n)}, \xi\right)(\xi j)^n\left[j + \frac{1}{2} + \frac{1}{8j}\right]. \end{aligned} \tag{46}$$

Thus, we have obtained (35). ∎

The more general result follows.

THEOREM 9. *It holds*

$$|E_{r,\xi}(f;x)| \le \left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1\left(f^{(n)},\xi\right), \tag{47}$$

all $x \in \mathbb{R}$, $\xi > 0$, *and furthermore,*

$$\|E_{r,\xi}f\|_\infty \le \left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1\left(f^{(n)},\xi\right), \qquad \xi > 0, \quad n \in \mathbb{N}. \tag{48}$$

PROOF. From (7), (34), and (35), we have

$$\begin{aligned}|E_{r,\xi}(f;x)| &\le \sum_{j=1}^{r}\binom{r}{j}\xi^n\left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1\left(f^{(n)},\xi\right)\\ &= \left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1\left(f^{(n)},\xi\right).\end{aligned} \tag{49}$$

That is proving (47). ∎

Some alternative basic results follow.

PROPOSITION 1. *All assumptions as above. Then,*

$$|P_{\xi j}(f;x) - f(x)| \le \sum_{m=1}^{\lfloor n/2\rfloor}\left|f^{(2m)}(x)\right|(\xi j)^{2m} + (\xi j)^n\left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1\left(f^{(n)},\xi\right) \tag{50}$$

and

$$|P_\xi(f;x) - f(x)| \le \sum_{m=1}^{\lfloor n/2\rfloor}\left|f^{(2m)}(x)\right|\xi^{2m} + \frac{13}{8}\xi^n\omega_1\left(f^{(n)},\xi\right), \qquad x \in \mathbb{R}, \quad \xi > 0, \quad n \in \mathbb{N}. \tag{51}$$

Assuming that $\|f^{(2m)}\|_\infty < \infty$, $m = 1, \dots, \lfloor n/2\rfloor$, *we obtain*

$$\|P_{\xi j}f - f\|_\infty \le \sum_{m=1}^{\lfloor n/2\rfloor}\left\|f^{(2m)}\right\|_\infty(\xi j)^{2m} + (\xi j)^n\left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1\left(f^{(n)},\xi\right) \tag{52}$$

and

$$\|P_\xi f - f\|_\infty \le \sum_{m=1}^{\lfloor n/2\rfloor}\left\|f^{(2m)}\right\|_\infty\xi^{2m} + \frac{13}{8}\xi^n\omega_1\left(f^{(n)},\xi\right), \qquad \xi > 0, \quad n \in \mathbb{N}. \tag{53}$$

PROOF. This is proven by (29) and (35), etc. ∎

We give the following.

COROLLARY 1. $n = 2$ CASE. *It holds*

$$|P_{\xi j}(f;x) - f(x)| \le (\xi j)^2\left[|f''(x)| + \left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1(f'',\xi)\right] \tag{54}$$

and

$$|P_\xi(f;x) - f(x)| \le \xi^2\left(|f''(x)| + \frac{13}{8}\omega_1(f'',\xi)\right), \qquad x \in \mathbb{R}, \quad \xi > 0. \tag{55}$$

Furthermore, when $\|f''\|_\infty < \infty$, *we have*

$$\|P_{\xi j}f - f\|_\infty \le (\xi j)^2\left[\|f''\|_\infty + \left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1(f'',\xi)\right] \tag{56}$$

and

$$\|P_\xi f - f\|_\infty \le \xi^2\left(\|f''\|_\infty + \frac{13}{8}\omega_1(f'',\xi)\right), \qquad \xi > 0. \tag{57}$$

PROOF. This is proven by Proposition 1. ∎

Corollary 2 follows.

COROLLARY 2. $n=1$ CASE. *It holds*

$$|P_{\xi j}(f;x)-f(x)| \leq \xi\left[j^2+\frac{j}{2}+\frac{1}{8}\right]\omega_1(f',\xi) \tag{58}$$

and

$$|P_{\xi}(f;x)-f(x)| \leq \frac{13}{8}\xi\omega_1(f',\xi), \qquad x\in\mathbb{R}, \quad \xi>0. \tag{59}$$

Furthermore, we have

$$\|P_{\xi j}f-f\|_{\infty} \leq \xi\left[j^2+\frac{j}{2}+\frac{1}{8}\right]\omega_1(f',\xi) \tag{60}$$

and

$$\|P_{\xi}f-f\|_{\infty} \leq \frac{13}{8}\xi\omega_1(f',\xi), \quad \xi>0. \tag{61}$$

PROOF. This is proven by proof of Theorem 8 for $n=1$, see also (29). ∎

More generally, we have the following.

PROPOSITION 2. *All assumptions as above. Then,*

$$\begin{aligned}|P_{r,\xi}(f;x)-f(x)| &\leq \sum_{j=1}^{r}\sum_{m=1}^{\lfloor n/2\rfloor}\left|f^{(2m)}(x)\right|\binom{r}{j}\frac{\xi^{2m}}{j^{n-2m}}\\ &+\left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1\left(f^{(n)},\xi\right),\end{aligned} \tag{62}$$

all $x\in\mathbb{R}$, $\xi>0$, $n\in\mathbb{N}$. Furthermore, by assuming that $\|f^{(2m)}\|_{\infty}<\infty$, for $m=1,\dots,\lfloor n/2\rfloor$, we obtain

$$\begin{aligned}\|P_{r,\xi}f-f\|_{\infty} &\leq \sum_{j=1}^{r}\sum_{m=1}^{\lfloor n/2\rfloor}\left\|f^{(2m)}\right\|_{\infty}\binom{r}{j}\frac{\xi^{2m}}{j^{n-2m}}\\ &+\left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1\left(f^{(n)},\xi\right), \qquad \xi>0, \quad n\in\mathbb{N}.\end{aligned} \tag{63}$$

PROOF. From (27) and (50), we have

$$\begin{aligned}|P_{r,\xi}(f;x)-f(x)| &\leq \sum_{j=1}^{r}\binom{r}{j}\left[\sum_{m=1}^{\lfloor n/2\rfloor}\left|f^{(2m)}(x)\right|\xi^{2m}j^{2m-n}+\xi^n\left[j+\frac{1}{2}+\frac{1}{8j}\right]\omega_1\left(f^{(n)},\xi\right)\right]\\ &=\sum_{j=1}^{r}\sum_{m=1}^{\lfloor n/2\rfloor}\left|f^{(2m)}(x)\right|\binom{r}{j}\frac{\xi^{2m}}{j^{n-2m}}\\ &+\left(\sum_{j=1}^{r}\binom{r}{j}\left[j+\frac{1}{2}+\frac{1}{8j}\right]\right)\xi^n\omega_1\left(f^{(n)},\xi\right).\end{aligned}$$ ∎

We have the following.

COROLLARY 3. $n=2$ CASE. *It holds*

$$|P_{r,\xi}(f;x)-f(x)| \leq \xi^2\left\{(2^r-1)|f''(x)|+\left(\sum_{j=1}^{r}\binom{r}{j}\left(j+\frac{1}{2}+\frac{1}{8j}\right)\right)\omega_1(f'',\xi)\right\}, \tag{64}$$

all $x \in \mathbb{R}$, $\xi > 0$. *Furthermore, by assuming that* $\|f''\|_\infty < \infty$, *we obtain*

$$\|P_{r,\xi} f - f\|_\infty \leq \xi^2 \{(2^r - 1) \|f''\|_\infty + \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \omega_1 (f'', \xi) \}, \qquad \xi > 0. \tag{65}$$

PROOF. This is proven by Proposition 2. ∎

We also give the following.

COROLLARY 4. $n = 1$ CASE. *It holds*

$$|P_{r,\xi}(f;x) - f(x)| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi \omega_1 (f', \xi), \tag{66}$$

all $x \in \mathbb{R}$, $\xi > 0$. *Furthermore, we have*

$$\|P_{r,\xi} f - f\|_\infty \leq \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j} \right) \right) \xi \omega_1 (f', \xi), \qquad \xi > 0. \tag{67}$$

PROOF. By use of (7), (27) and (58). ∎

Next, we present simultaneous approximation results of P_ξ to I with rates.

THEOREM 10. *Let* $f \in C^{n+k}(\mathbb{R})$, $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$, *and* $\omega_1(f^{(n+i)}, h) < \infty$, $h > 0$, *for* $i = 0, 1, \dots, k$. *We consider the assumptions of Theorem 4 as valid for* $n = k$ *there. Then,*

(1)
$$\left| (\Delta_\xi (f;x))^{(i)} \right| \leq \frac{13}{8} \xi^n \omega_1 \left(f^{(n+i)}, \xi \right), \tag{68}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$, $n \in \mathbb{N}$,

(2)
$$\left| (P_\xi (f;x))^{(i)} - f^{(i)}(x) \right| \leq \sum_{m=1}^{\lfloor n/2 \rfloor} \left| f^{(2m+i)}(x) \right| \xi^{2m} + \frac{13}{8} \xi^n \omega_1 \left(f^{(n+i)}, \xi \right), \tag{69}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$, $n \in \mathbb{N}$,

(3) $n = 2$ *case,*
$$\left| (P_\xi (f;x))^{(i)} - f^{(i)}(x) \right| \leq \xi^2 \left(\left| f^{(2+i)}(x) \right| + \frac{13}{8} \omega_1 \left(f^{(2+i)}, \xi \right) \right), \tag{70}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$, *and*

(4) $n = 1$ *case,*
$$\left| (P_\xi (f;x))^{(i)} - f^{(i)}(x) \right| \leq \frac{13}{8} \xi \omega_1 \left(f^{(1+i)}, \xi \right), \tag{71}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$.

PROOF. By using Theorem 4, Theorem 8, equation (36), Proposition 1, equation (51), Corollary 1, equation (55), and Corollary 2, equation (59). ∎

We finish with operator $P_{r,\xi}$ simultaneous approximation results to I with rates.

THEOREM 11. *Let $f \in C^{n+k}(\mathbb{R})$, $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$, and $\omega_1(f^{(n+i)}, h) < \infty$, $h > 0$, for $i = 0, 1, \dots, k$. We consider the assumptions of Theorem 5 as valid for $n = k$ there. Then,*

(1)
$$\left|(E_{r,\xi}(f;x))^{(i)}\right| \le \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j}\right)\right) \xi^n \omega_1\left(f^{(n+i)}, \xi\right), \tag{72}$$

for all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$, $n \in \mathbb{N}$,

(2)
$$\begin{aligned} \left|(P_{r,\xi}(f;x))^{(i)} - f^{(i)}(x)\right| &\le \sum_{j=1}^{r} \sum_{m=1}^{\lfloor n/2 \rfloor} \left|f^{(2m+i)}(x)\right| \binom{r}{j} \frac{\xi^{2m}}{j^{n-2m}} \\ &+ \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j}\right)\right) \xi^n \omega_1\left(f^{(n+i)}, \xi\right), \end{aligned} \tag{73}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$, $n \in \mathbb{N}$,

(3) *$n = 2$ case,*

$$\begin{aligned} \left|(P_{r,\xi}(f;x))^{(i)} - f^{(i)}(x)\right| &\le \xi^2 \left\{(2^r - 1)\left|f^{(2+i)}(x)\right| \right. \\ &\left. + \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j}\right)\right) \omega_1\left(f^{(2+i)}, \xi\right)\right\}, \end{aligned} \tag{74}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$, and

(4) *$n = 1$ case,*

$$\left|(P_{r,\xi}(f;x))^{(i)} - f^{(i)}(x)\right| \le \left(\sum_{j=1}^{r} \binom{r}{j} \left(j + \frac{1}{2} + \frac{1}{8j}\right)\right) \xi \omega_1\left(f^{(1+i)}, \xi\right), \tag{75}$$

all $x \in \mathbb{R}$, $\xi > 0$, $i = 0, 1, \dots, k$.

PROOF. By using Theorem 5, Theorem 9 (47), Proposition 2 (62), Corollary 3 (64), and Corollary 4 (66). ∎

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